CLASSIFICATION OF QUASIFINITE $\mathcal{W}_\infty$-MODULES*

BY

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ABSTRACT

It is proved that an irreducible quasifinite $\mathcal{W}_\infty$-module is a highest or lowest weight module or a module of the intermediate series; a uniformly bounded indecomposable weight $\mathcal{W}_\infty$-module is a module of the intermediate series. For a nondegenerate additive subgroup Γ of $\mathbb{F}^n$, where $\mathbb{F}$ is a field of characteristic zero, there is a simple Lie or associative algebra $\mathcal{W}(\Gamma, n)^{(1)}$ spanned by differential operators $uD_1^{m_1} \cdots D_n^{m_n}$ for $u \in \mathbb{F}[\Gamma]$ (the group algebra), and $m_i \geq 0$ with $\sum_{i=1}^n m_i \geq 1$, where D_i are degree operators. It is also proved that an indecomposable quasifinite weight $\mathcal{W}(\Gamma, n)^{(1)}$-module is a module of the intermediate series if Γ is not isomorphic to $\mathbb{Z}$.

1. Introduction

Let us start with the general definition. For an algebraically closed field $\mathbb{F}$ of characteristic zero, let Γ be a **nondegenerate** additive subgroup of $\mathbb{F}^n$, i.e., it contains an $\mathbb{F}$-basis of $\mathbb{F}^n$. Let $\mathbb{F}[\Gamma] = \mathrm{span}\{t^\alpha | \alpha \in \Gamma\}$ denote the group algebra of Γ with the algebraic operation $t^\alpha \cdot t^\beta = t^{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$. We define the **degree operators** D_i to be the derivations of $\mathbb{F}[\Gamma]$ determined by $D_i \colon t^\alpha \mapsto \alpha_i t^\alpha$ for $\alpha \in \Gamma, i = 1, \ldots, n$. Here and below, an element $\alpha \in \mathbb{F}^n$ is always written as $\alpha = (\alpha_1, \ldots, \alpha_n)$. The **Lie algebra** $\mathcal{W}(\Gamma, n)$ **of Weyl type**

[S4] is a tensor product space of the group algebra $\mathbb{F}[\Gamma]$ with the polynomial algebra $\mathbb{F}[D_1, \ldots, D_n]$:

$$(1.1) \qquad \mathcal{W}(\Gamma, n) = \mathbb{F}[\Gamma] \otimes \mathbb{F}[D_1, \ldots, D_n] = \mathrm{span}\{t^\alpha D^\mu | \alpha \in \Gamma, \mu \in \mathbb{Z}_+^n\},$$

where $D^\mu = \prod_{i=1}^n D_i^{\mu_i}$, with the Lie bracket:

$$[t^\alpha D^\mu, t^\beta D^\nu] = (t^\alpha D^\mu) \cdot (t^\beta D^\nu) - (t^\beta D^\nu) \cdot (t^\alpha D^\mu),$$

and

$$(1.2) \qquad (t^\alpha D^\mu) \cdot (t^\beta D^\nu) = \sum_{\lambda \in \mathbb{Z}_+^n} \binom{\mu}{\lambda} \beta^\lambda t^{\alpha+\beta} D^{\mu+\nu-\lambda},$$

where $\beta^\lambda = \prod_{i=1}^n \beta_i^{\lambda_i}$ (here without confusion, we use notation β^λ similar to notation D^μ in (1.1)), and $\binom{\mu}{\lambda} = \prod_{i=1}^n \binom{\mu_i}{\lambda_i}$. Furthermore, for $i, j \in \mathbb{F}$, $\binom{i}{j} = i(i-1)\cdots(i-j+1)/j!$ if $j \in \mathbb{Z}_+$, or $\binom{i}{j} = 0$ otherwise.

It is proved [S3] that $\mathcal{W}(\Gamma, n)$ has a nontrivial universal central extension if and only if $n = 1$. The Lie bracket for the universal central extension $\widehat{\mathcal{W}}(\Gamma, 1)$ of $\mathcal{W}(\Gamma, 1)$ is defined by

$$
\begin{aligned}
(1.3) \quad [t^\alpha [D]_\mu, t^\beta [D]_\nu] = & (t^\alpha [D]_\mu) \cdot (t^\beta [D]_\nu) - (t^\beta [D]_\nu) \cdot (t^\alpha [D]_\mu) \\
& + \delta_{\alpha, -\beta} (-1)^\mu \mu! \nu! \binom{\alpha + \mu}{\mu + \nu + 1} C,
\end{aligned}
$$

for $\alpha, \beta \in \Gamma \subset \mathbb{F}, \mu, \nu \in \mathbb{Z}_+$, where $[D]_\mu = D(D-1)\cdots(D-\mu+1)$, and C is a central element of $\widehat{\mathcal{W}}(\Gamma, 1)$. The 2-cocycle of $\mathcal{W}(\mathbb{Z}, 1)$ corresponding to (1.3) seems to appear first in [KP].

Denote by $\mathcal{W}(\Gamma, n)^{(1)}$ the Lie subalgebra of $\mathcal{W}(\Gamma, n)$ spanned by $\{t^\alpha D^\mu | \alpha \in \Gamma, |\mu| \geq 1\}$, where $|\mu| = \sum_{i=1}^n \mu_i$. Similarly, we can define $\widehat{\mathcal{W}}(\Gamma, 1)^{(1)}$. Then $\mathcal{W}_{1+\infty} = \widehat{\mathcal{W}}(\mathbb{Z}, 1)$ and $\mathcal{W}_\infty = \widehat{\mathcal{W}}(\mathbb{Z}, 1)^{(1)}$ are the well-known $\mathcal{W}$-infinity algebras, which arise naturally in various physical theories such as conformal field theory, the theory of the quantum Hall effect, etc. and which receive intensive studies in the literature (cf. [BKLY, FKRW, KL, KR1, KR2, KWY, S4]).

Note that $\mathcal{W}(\Gamma, n)^{(1)}$ is also an associative algebra under the product (1.2). It can be proved that $\mathcal{W}(\Gamma, n)^{(1)}$ is simple as a Lie or associative algebra (cf. [SZ1]). We denote it by $\mathcal{A}(\Gamma, n)^{(1)}$ when we consider it as an associative algebra. Clearly an $\mathcal{A}(\Gamma, n)^{(1)}$-module is also a $\mathcal{W}(\Gamma, n)^{(1)}$-module, but not necessarily the converse. Thus it suffices to consider $\mathcal{W}(\Gamma, n)^{(1)}$-modules. The Lie algebra $\mathcal{W}(\Gamma, n)^{(1)} = \bigoplus_{\alpha \in \Gamma} \mathcal{W}(\Gamma, n)_\alpha^{(1)}$ is Γ-graded with the grading space

$$(1.4) \qquad \mathcal{W}(\Gamma, n)_\alpha^{(1)} = \mathrm{span}\{t^\alpha D^\mu | \mu \in \mathbb{Z}_+^n \setminus \{0\}\} \quad \text{for } \alpha \in \Gamma.$$

In [S4], one of us classified the quasifinite modules over $\mathcal{W}(\Gamma, n)$. In this paper, we shall consider the more difficult problem of classifying the quasifinite modules over $\mathcal{W}(\Gamma, n)^{(1)}$. Here, a $\mathcal{W}(\Gamma, n)^{(1)}$-module V is called a **quasifinite module** if $V = \bigoplus_{\alpha \in \Gamma} V_\alpha$ is a Γ-graded $\mathbb{F}$-vector space such that $\mathcal{W}(\Gamma, n)^{(1)}_\alpha V_\beta \subset V_{\alpha+\beta}$, $\dim V_\alpha < \infty$ for $\alpha, \beta \in \Gamma$. When we study the representations of Lie algebras of this kind, since each grading space in (1.4) is still infinite-dimensional, the classification of quasifinite modules is thus a nontrivial problem, as pointed in [KL].

For $\alpha \in \mathbb{F}^n$, one can define quasifinite $\mathcal{W}(\Gamma, n)^{(1)}$- or $\widehat{\mathcal{W}}(\Gamma, 1)^{(1)}$-modules A_α, B_α as follows: They have basis $\{y_\beta | \beta \in \Gamma\}$ such that the central element C acts trivially and

$$A_\alpha : (t^\beta D^\mu) y_\gamma = (\alpha + \gamma)^\mu y_{\beta+\gamma},$$
$$B_\alpha : (t^\beta D^\mu) y_\gamma = (-1)^{|\mu|+1} (\alpha + \beta + \gamma)^\mu y_{\beta+\gamma},$$

for $\beta, \gamma \in \Gamma, \mu \in \mathbb{Z}_+^n \backslash \{0\}$ (where $(\alpha + \gamma)^\mu$ is a notation as β^λ in (1.2)). These modules are defined in [S4, Z]. Obviously, A_α or B_α is irreducible if and only if $\alpha \notin \Gamma$. Clearly A_α is also an $\mathcal{A}(\Gamma, n)^{(1)}$-module, but not B_α. We refer to any subquotient module of A_α or B_α as a **module of the intermediate series** (cf. [S4]). Then the main result of the present paper is the following.

THEOREM 1.1: (i) *An irreducible quasifinite module over $\mathcal{W}(\mathbb{Z}, 1)^{(1)}$ or over $\mathcal{W}_\infty = \widehat{\mathcal{W}}(\mathbb{Z}, 1)^{(1)}$ is a highest or lowest weight module, or a module of the intermediate series.*

(ii) *An irreducible quasifinite $\mathcal{W}(\Gamma, n)^{(1)}$- or $\widehat{\mathcal{W}}(\Gamma, 1)^{(1)}$-module is a module of the intermediate series if Γ is not isomorphic to $\mathbb{Z}$.*

Since the complete description of irreducible quasifinite highest weight modules was obtained in [KL] and lowest weight modules are dual of highest weight modules, Theorem 1.1 and results in [KL] in fact give a complete classification of irreducible quasifinite modules. Theorem 1.1 also gives a classification of irreducible quasifinite modules over the associative algebras $\mathcal{A}(\Gamma, n)^{(1)}$.

The analogous results to the above theorem for affine Lie algebras, the Virasoro algebra, higher rank Virasoro algebras and Lie algebras of Weyl type or Block type have been obtained in [C, M, S4, S5, S6] (also, cf. [S2]).

A quasifinite module V is **uniformly bounded** if there exists $N \geq 0$ such that $\dim V_\beta \leq N$ for all $\beta \in \Gamma$; it is called a **weight module** if $D_1, \ldots, D_n$ are semi-simple operators on V.

THEOREM 1.2: (i) *A uniformly bounded indecomposable weight* $\mathcal{W}(\mathbb{Z},1)^{(1)}$- *or* $\mathcal{W}_\infty$-*module is a module of the intermediate series.*

(ii) *A quasifinite indecomposable weight* $\mathcal{W}(\Gamma,n)^{(1)}$- *or* $\widehat{\mathcal{W}}(\Gamma,n)^{(1)}$-*module is a module of the intermediate series if* Γ *is not isomorphic to* $\mathbb{Z}$.

Finally we would like to point out that although the main result of the present paper is similar to that of [S4], one can see below that the proof is more technical than that of [S4] due to the fact that the elements $t^\beta = t^\beta D^0, \beta \in \Gamma$, do not appear in $\mathcal{W}(\Gamma,n)^{(1)}$.

2. Quasifinite $\mathcal{W}_\infty$-modules

First we prove Theorem 1.1(i) and Theorem 1.2(i). We shall only work on the non-central extension case since the proof of the central extension case is similar.

Now consider the Lie algebra

$$W := \mathcal{W}(\mathbb{Z},1)^{(1)} = \mathrm{span}\{t^i D^j | i \in \mathbb{Z}, j \in \mathbb{Z}_+\backslash\{0\}\}.$$

In this case $D = t\frac{d}{dt}$, and by (1.4), $W = \bigoplus_{i\in\mathbb{Z}} W_i$ is $\mathbb{Z}$-graded with

$$W_i = \mathrm{span}\{t^i D^j | j \in \mathbb{Z}_+\backslash\{0\}\} = \{t^i D f(D) | f(D) \in \mathbb{F}[D]\}$$

for $i \in \mathbb{Z}$. By (1.2), we have

$$(2.1) \quad \begin{aligned} &[t^i D f(D), t^j D g(D)] \\ &= t^{i+j} D((D+j)f(D+j)g(D) - (D+i)g(D+i)f(D)), \end{aligned}$$

for $i, j \in \mathbb{Z}, f(D), g(D) \in \mathbb{F}[D]$. Also, W has a triangular decomposition $W = W_+ \oplus W_0 \oplus W_-$, where in general, for any $\mathbb{Z}$-graded space M, we always use notations M_+, M_-, M_0 and $M_{[p,q)}$ to denote the subspaces spanned by elements of degree k with $k > 0, k < 0, k = 0$ and $p \le k < q$ respectively. Denote $\mathrm{Vir} = \bigoplus_{i\in\mathbb{Z}} \mathbb{F}t^i D$, which is the (centerless) Virasoro algebra.

LEMMA 2.1: *Let* S *be a subspace of* W_0 *with finite co-dimension. Given* $i_0 > 0$, *let* $M_{i_0,S}$ *denote the subalgebra of* W *generated by* $t^{i_0} D, t^{i_0+1} D, t^{i_0} D^2$ *and* S. *Then there exists some integer* $K > 0$ *such that* $W_{[K,\infty)} \subset M_{i_0,S}$.

Proof: By the assumption of S, there exists some integer $m_0 \ge 0$ such that for all integer $m \ge m_0$, there exists a polynomial $Df(D) \in S$ with $\deg f = m$. We shall prove by induction on m the following claim.

CLAIM 1: *For any $m \in \mathbb{Z}$ with $1 \le m \le m_0$, there exists some integer $K_m > mK_{m-1}$ (where we take $K_0 = i_0$) such that $t^k D^m \in M_{i_0,S}$ for all integers $k \ge K_m$.*

Suppose $m = 1$. For any integer k sufficiently large enough, we can write $k = k_1 i_0 + k_2 (i_0 + 1)$ for some $k_1, k_2 \in \mathbb{Z}_+\backslash\{0\}$, so $t^k D$ can be generated by $t^{i_0} D, t^{i_0+1} D$, i.e., $t^k D \in M_{i_0,S}$. Thus we can take some integer $K_1 > i_0$ large enough to ensure that the claim holds for $m = 1$. Suppose $1 < m \le m_0$ and inductively assume that the claim holds for $m - 1$. Take $K_m = m K_{m-1} + i_0$. Then for any $k \ge K_m$, by (2.1) we have $at^k D^m \equiv [t^{k-i_0} D^{m-1}, t^{i_0} D^2] \equiv 0 \pmod{M_{i_0,S}}$, where $a = ((m-1)i_0 - 2(k - i_0)) < 0$, i.e., $t^k D^m \in M_{i_0,S}$. Thus the claim holds for m.

Now take $K = K_{m_0}$. For any integer $k \ge K$, we can now prove by induction on $m \ge 1$ that $t^k D^m \in M_{i_0,S}$ as follows: If $m \le m_0$, this immediately follows from Claim 1. Assume that $m > m_0$. Let $f(D)$ be a polynomial of degree $m-1 \ge m_0$ such that $Df(D) \in S$, then by (2.1), $kmt^k D^m \equiv [t^k D, Df(D)] \equiv 0 \pmod{M_{i_0,S}}$. This proves that $W_{[K,\infty)} \subset M_{i_0,S}$. ∎

LEMMA 2.2: *Assume that V is an irreducible quasifinite W-module without a highest or lowest weight. For any $i, j \in \mathbb{Z}, i \ne 0, -1$, the linear map*

$$t^i D|_{V_j} \oplus t^{i+1} D|_{V_j} \oplus t^i D^2|_{V_j}: V_j \to V_{i+j} \oplus V_{i+j+1} \oplus V_{i+j}$$

is injective. In particular, $\dim V_j \le 2(\dim V_0) + \dim V_1$ for $j \in \mathbb{Z}$.

Proof (cf. [S4]): Being irreducible, V must be a weight module, i.e., there exists some $\alpha \in \mathbb{F}$, such that

$$(2.2) \qquad V_i = \{v \in V | Dv = (\alpha + i)v\}.$$

Say, $i > 0$ and $(t^i D)v_0 = (t^{i+1}D)v_0 = (t^i D^2)v_0 = 0$ for some $0 \ne v_0 \in V_j$. By shifting the grading index of V_j if necessary, we can suppose $j = 0$. Let S be the kernel of the linear map $W_0 \to \text{End}(V_0): D^m \mapsto D^m|_{V_0}$ for $m \ge 1$. Since $\dim V_0 < \infty$, S is a subspace of W_0 with finite co-dimension. Then $M_{i,S}v_0 = 0$ and by Lemma 2.1, we have $W_{[K,\infty)}v_0 = 0$ for some $K > 0$.

For any subspace M of W, we use $U(M)$ to denote the subspace, which is the span of the standard monomials with respect to a basis of M, of the universal enveloping algebra of W. Since $W = W_{[1,K)} + W_0 + W_- + W_{[K,\infty)}$, using the PBW theorem and the irreducibility of V, we have

$$(2.3) \qquad \begin{aligned} V &= U(W)v_0 = U(W_{[1,K)})U(W_0 + W_-)U(W_{[K,\infty)})v_0 \\ &= U(W_{[1,K)})U(W_0 + W_-)v_0. \end{aligned}$$

Note that V_+ is a W_+-module. Let V'_+ be the W_+-submodule of V_+ generated by $V_{[0,K)}$. We want to prove that $V_+ = V'_+$.

So let $k \geq 0$ and let $x \in V_+$ have degree $\deg x = k$. If $0 \leq k < K$, then by definition, $x \in V'_+$. Suppose $k \geq K$. Using (2.3), x is a linear combination of the form $u_1 x_1$ with $u_1 \in W_{[1,K)}, x_1 \in V$. Thus the degree $\deg u_1$ of u_1 satisfies $1 \leq \deg u_1 < K$, so $0 < \deg x_1 = k - \deg u_1 < k$. By inductive hypothesis, $x_1 \in V'_+$, and thus $x \in V'_+$. This proves that $V_+ = V'_+$.

The fact that $V_+ = V'_+$ means that the W_+-module V_+ is generated by the finite dimensional space $V_{[0,K)}$. Choose a basis B of $V_{[0,K)}$. Then for any $x \in B$, we have $x = u_x v_0$ for some $u_x \in U(W)$. Regarding u_x as a polynomial with respect to a basis of W, by induction on the polynomial degree and using the formula $[w, w_1 w_2] = [w, w_1]w_2 + w_1[w, w_2]$ for $w \in W$, $w_1, w_2 \in U(W)$, we see that there exists a positive integer $k_x > K$ sufficiently large enough such that $[W_{[k_x,\infty)}, u_x] \subset U(W)W_{[K,\infty)}$. Then from $W_{[K,\infty)}v_0 = 0$, we have $W_{[k_x,\infty)}x = [W_{[k_x,\infty)}, u_x]v_0 + u_x W_{[k_x,\infty)}v_0 = 0$. Take $K' = \max\{k_x | x \in B\}$, then $W_{[K',\infty)}V_{[0,K)} = 0$ and

$$W_{[K',\infty)}V_+ = W_{[K',\infty)}U(W_+)V_{[0,K)} = U(W_+)W_{[K',\infty)}V_{[0,K)} = 0.$$

Since there exists some integer $K_1 > K'$ sufficiently large enough to ensure that $W_+ \subset W_{[K',\infty)} + [W_{[-K_1,0)}, W_{[K',\infty)}]$, this means that we have $W_+ V_{[K_1,\infty)} = 0$. Now suppose $x \in V_{[K_1+K,\infty)}$. Then by (2.3), it is a sum of elements of the form $u_1 x_1$ such that $u_1 \in W_{[1,K)}$. But then x_1 has degree $\deg x_1 > \deg x - K \geq K_1$, so $x_1 \in V_{[K_1,\infty)}$. Thus from $W_+ V_{[K_1,\infty)} = 0$, we have $u_1 x_1 = 0$, i.e., $x = 0$. This proves that V has no degree $\geq K_1 + K$.

Now let K'' be the maximal integer such that $V_{K''} \neq 0$. Since W_0 is commutative, there exists a common eigenvector $v'_0 \in V_{K''}$ for W_0. Then v'_0 is a highest weight vector of W, this contradicts the assumption of the lemma. ∎

Theorem 1.1(i) will follow from Theorem 1.2(i) and Lemma 2.2, so it suffices to prove Theorem 1.2(i). Thus from now on, we suppose V is a uniformly bounded indecomposable weight W-module such that (2.2) holds.

Regarding V as a weight module over the Virasoro algebra Vir, by [S2], there exists some $N \geq 0$ such that $\dim V_k = N$ for all $k \in \mathbb{Z}$ with $k + \alpha \neq 0$, where $\alpha \in \mathbb{F}$ is fixed such that (2.2) holds, and V has only a finite composition factors as a Vir-module, and $t^{-1}D|_{V_k}: V_k \to V_{k-1}$ is bijective when $k >> 0$. So, we can find a basis $Y_k = (y_k^{(1)}, \ldots, y_k^{(N)})$ of V_k such that

(2.4) $(t^{-1}D)Y_k = Y_{k-1}$ for $k >> 0$.

We shall assume that $N \geq 1$ since the proof is trivial if $N = 0$. In the following, we always suppose that k is an integer such that $k >> 0$. Assume that

$$(t^i D)Y_k = Y_{k+i}P_{i,k} \quad \text{for some } N \times N \text{ matrices } P_{i,k} \text{ and } i \in \mathbb{Z}.$$

By (2.2), (2.4) and applying $[t^{-1}D, t^i D] = (i+1)t^{i-1}D$ to Y_k for $i = 1, 2$, we obtain

$$(2.5) \qquad P_{-1,K} = 1, P_{0,k} = \bar{k}, P_{1,k} = [\bar{k}]^2 + P_1, P_{2,k} = [\bar{k}]^3 + 3\bar{k}P_1 + P_2,$$

for some $N \times N$ matrices P_1, P_2. Here and below, for convenience, we always identify a scalar $a \in \mathbb{F}$ with the corresponding $N \times N$ scalar matrix $a \cdot \mathbf{1}_N$ when the context is clear, where $\mathbf{1}_N$ is the $N \times N$ identity matrix. We also denote $\bar{k} = k + \alpha$ for $k \in \mathbb{Z}$, and in general, we use the notation $[a]^j = a(a+1)\cdots(a+j-1)$ for $a \in \mathbb{F}$, $j \in \mathbb{Z}_+$ (cf. notation $[D]_j$ in (1.3)). By choosing a composition series of V regarding as a Vir-module, we can suppose P_1, P_2 are upper-triangular matrices. Applying $[tD, t^2 D] = t^3 D$ to Y_k, by (2.5), we obtain

$$(2.6) \qquad P_{3,k} = [\bar{k}]^4 + 6[\bar{k}]^2 P_1 + 4\bar{k}P_2 + P_3,$$

where $P_3 = -3(2P_1 + P_1^2 - 2P_2) + [P_1, P_2]$, and $[P_1, P_2] = P_1P_2 - P_2P_1$ is the usual Lie bracket. Recall that $D = t\frac{d}{dt}$. From this, one has $t^{i+j}(\frac{d}{dt})^j = t^i[D]_j$ for $i \in \mathbb{Z}, j \in \mathbb{Z}_+\backslash\{0\}$. In the following, we shall often use notation $\frac{d}{dt}$ instead of D whenever it is convenient. Remember that $\frac{d}{dt}$ is an operator of degree -1. Assume that

$$\left(\frac{d}{dt}\right)^i Y_k = Y_{k-i}Q_{i,k} \quad \text{for some } N \times N \text{ matrices } Q_{i,k} \text{ and } i \geq 1.$$

Using $[\frac{d}{dt}, (\frac{d}{dt})^i] = 0$, we obtain that $Q_{i,k} = Q_i$ which does not depend on k. Note that since $\frac{d}{dt} = t^{-1}D$, we have $Q_1 = \mathbf{1}_N$ by (2.4).

LEMMA 2.3: P_1, P_2 and $Q_i - \mathbf{1}_N$ are strict upper-triangular matrices for all $i \in 2\mathbb{Z}_+ + 1$.

Proof: So assume that $N > 1$. By (2.1) or (1.2), we can deduce that

$$-[i+1]_4\left(\frac{d}{dt}\right)^{i-2} = 3\left[t^2\frac{d}{dt}, \left[t^2\frac{d}{dt}, \left(\frac{d}{dt}\right)^i\right]\right] + 2(2i-1)\left[t^3\frac{d}{dt}, \left(\frac{d}{dt}\right)^i\right],$$

$$0 = \left[t^2\frac{d}{dt}, \left[t^2\frac{d}{dt}, \left[t^2\frac{d}{dt}, \left(\frac{d}{dt}\right)^i\right]\right]\right]$$

$$+ (i-1)(i-2)\left[t^4\frac{d}{dt}, \left(\frac{d}{dt}\right)^i\right] + 2(i-1)\left[t^2\frac{d}{dt}, \left[t^3\frac{d}{dt}, \left(\frac{d}{dt}\right)^i\right]\right],$$

$$[i+1]_6\Big(\frac{d}{dt}\Big)^{i-4} = 10\Big[t^3\frac{d}{dt},\Big[t^3\frac{d}{dt},\Big(\frac{d}{dt}\Big)^i\Big]\Big] - 6(i-4)\Big[t^5\frac{d}{dt},\Big(\frac{d}{dt}\Big)^i\Big]$$
$$- 15\Big[t^2\frac{d}{dt},\Big[t^4\frac{d}{dt},\Big(\frac{d}{dt}\Big)^i\Big]\Big],$$

for $i \geq 1$, where in general $[a]_j$ is a notation similar to $[D]_j$ in (1.3) (cf. notation $[a]^j$ in (2.5)). Here and below, we make the convention that if a notion is not defined but technically appears in an expression, we always treat it as zero; for instance, $(\frac{d}{dt})^{i-2} = 0$ if $i \leq 2$. Applying these three formulas to Y_k, we obtain

$$-[i+1]_4Q_{i-2} = 3(P_{1,k-i+1}P_{1,k-i}Q_i - 2P_{1,k-i+1}Q_iP_{1,k} + Q_iP_{1,k+1}P_{1,k})$$
$$(2.7) \qquad\qquad + 2(2i-1)(P_{2,k-i}Q_i - Q_iP_{2,k}),$$

$$0 = P_{1,k-i+2}P_{1,k-i+1}P_{1,k-i}Q_i - 3P_{1,k-i+2}P_{1,k-i+1}Q_iP_{1,k}$$
$$+ 3P_{1,k-i+2}Q_iP_{1,k+1}P_{1,k} - Q_iP_{1,k+2}P_{1,k+1}P_{1,k}$$
$$+ (i-1)(i-2)(P_{3,k-i}Q_i - Q_iP_{3,k})$$
$$+ 2(i-1)(P_{1,k-i+2}(P_{2,k-i}Q_i - Q_iP_{2,k})$$
$$(2.8) \qquad\qquad - (P_{2,k-i+1}Q_i - Q_iP_{2,k+1})P_{1,k}),$$

$$[i+1]_6Q_{i-4} = 10(P_{2,k-i+2}P_{2,k-i}Q_i - 2P_{2,k-i+2}Q_iP_{2,k} + Q_iP_{2,k+2}P_{2,k})$$
$$- 6(i-4)(P_{4,k-i}Q_i - Q_iP_{4,k})$$
$$- 15(P_{1,k-i+3}(P_{3,k-i}Q_i - Q_iP_{3,k})$$
$$(2.9) \qquad\qquad - (P_{3,k-i+1}Q_i - Q_iP_{3,k+1})P_{1,k}),$$

for $i \geq 1$. We shall denote by $p_{i,k}^{(a,b)}$ the (a,b)-entry of the matrix $P_{i,k}$ and the like for other matrices. For a given position (a,b) with $1 \leq b \leq a \leq N$, suppose inductively we have proved

$$(2.10) \qquad\qquad q_i^{(a_1,b_1)} = 0$$

for all $i \in 2\mathbb{Z}_+ + 1$ and for $a_1 > a, b_1 \leq b$ or $a_1 \geq a, b_1 < b$. Now for convenience, we denote

$$p_{j,k} = p_{j,k}^{(a,a)}, \quad p'_{j,k} = p_{j,k}^{(b,b)}, \quad p_j = p_j^{(a,a)}, \quad p'_j = p_j^{(b,b)}, \quad q_j = q_j^{(a,b)}$$

for $j \in \mathbb{Z}$. Assume that $i \in 2\mathbb{Z}_+ + 1$. Using (2.10), by comparing the (a,b)-entries in (2.7)–(2.9), we obtain

$$-[i+1]_4q_{i-2} = (3(p_{1,k-i+1}p_{1,k-i} - 2p_{1,k-i+1}p'_{1,k} + p'_{1,k+1}p'_{1,k})$$
$$(2.11) \qquad\qquad + 2(2i-1)(p_{2,k-i} - p'_{2,k}))q_i,$$

$$0 = (p_{1,k-i+2}p_{1,k-i+1}p_{1,k-i} - 3p_{1,k-i+2}p_{1,k-i+1}p'_{1,k}$$

$$+ 3p_{1,k-i+2}p'_{1,k+1}p'_{1,k} - p'_{1,k+2}p'_{1,k+1}p'_{1,k}$$
$$+ (i-1)(i-2)(p_{3,k-i} - p'_{3,k})$$
$$+ 2(i-1)(p_{1,k-i+2}(p_{2,k-i} - p'_{2,k})$$

(2.12)
$$- (p_{2,k-i+1} - p'_{2,k+1})p'_{1,k}))q_i,$$

$$[i+1]_6 q_{i-4} = (10(p_{2,k-i+2}p_{2,k-i} - 2p_{2,k-i+2}p'_{2,k} + p'_{2,k+2}p'_{2,k})$$
$$- 6(i-4)(p_{4,k-i} - p'_{4,k}) - 15(p_{1,k-i+3}(p_{3,k-i} - p'_{3,k})$$

(2.13)
$$- (p_{3,k-i+1} - p'_{3,k+1})p'_{1,k}))q_i.$$

Applying $[t^2 \frac{d}{dt}, t^{j+1}\frac{d}{dt}] = (j-1)t^{i+2}\frac{d}{dt}$ to $y_k^{(b)}$ for $j = 4, 5$, since P_1, P_2 are upper-triangular matrices, using (2.5) and (2.6), we obtain

(2.14)
$$\begin{cases} p_{4,k} = [\overline{k}]^5 + 10[\overline{k}]^3 p_1 + 10[\overline{k}]^2 p_2 + 5\overline{k}p_3 + p_4, \\ p_{5,k} = [\overline{k}]^6 + 15[\overline{k}]^4 p_1 + 20[\overline{k}]^3 p_2 + 15[\overline{k}]^2 p_3 + 6\overline{k}p_4 + p_5, \end{cases}$$

where
$$p_4 = -2(24p_1 + 12p_1^2 - 18p_2 + p_1p_2),$$
$$p_5 = 5(-72p_1 - 34p_1^2 + p_1^3 + 48p_2 - 6p_1p_2).$$

We have similar formulas for $p'_{j,k}, j = 4, 5$. Applying $[t^3\frac{d}{dt}, t^4\frac{d}{dt}] = t^6\frac{d}{dt}$ to $y_k^{(b)}$, we obtain the following relation between p_1 and p_2, which is a well-known relation for the Virasoro algebra (cf. [S1]):

(2.15)
$$8p_1^2 + 4p_1^3 - 6p_1p_2 + p_2^2 = 0.$$

First we make the following assumption:

(2.16)
$$q_i \neq 0 \quad \text{for some } i \in 2\mathbb{Z}_+ + 1.$$

By replacing i by $i + 2$ in (2.11), since $[i+3]_4 \neq 0$ for $i \in 2\mathbb{Z}_+ + 1$, we see that (2.16) holds for infinite many $i \in 2\mathbb{Z}_+ + 1$. For fixed k, we denote by $f_1(i), f_2(i), f_3(i)$ the coefficients of q_i in (2.11)–(2.13) respectively. They are polynomials on i. Then (2.12) and (2.16) show that $f_2(i) = 0$ for infinite many i. Hence $f_2(i) = 0$ for all i. Using (2.5) and (2.6) in (2.12), it is straightforward to compute that the coefficient of i^4 in $f_2(i)$ is $p_1 - p'_1$. Therefore, $p'_1 = p_1$. Similarly, (2.11) and (2.13) show that

$$g(i) := [i+1]_4[i-1]_4 f_3(i) - [i+1]_6 f_1(i-2)f_1(i)$$

is zero for all i. It is a little lengthy but straightforward to compute that coefficient of i^{12} in $g(i)$ is $6p_1$ (using $p'_1 = p_1$). Thus $p_1 = 0$. By (2.15), $p_2 = 0$.

Thus also $p'_1 = p'_2 = 0$. Then (2.11) becomes $[i + 1]_4(q_i - q_{i-2}) = 0$. From this we obtain that $q_i = q_1$ for all $i \in 2\mathbb{Z}_+ + 1$.

Now we consider two cases: First assume that $b < a$. Then $q_1 := q_1^{(a,b)} = 0$ (recall that $Q_1 = 1_N$). If (2.16) holds, then the above in particular proves that $q_i = q_1 = 0$ for all $i \in 2\mathbb{Z}_+ + 1$. This contradicts (2.16). Thus (2.16) cannot hold for any i, i.e., in this case we have $q_i = 0$ for all $i \in 2\mathbb{Z}_+ + 1$.

Next assume that $a = b$. Then $q_1 := q_1^{(a,a)} = 1$ and so (2.16) holds for at least $i = 1$. Thus the above proves that $p_1 = p_2 = 0, q_i = q_1$, i.e., in this case we have $p_1^{(a,a)} = p_2^{(a,a)} = 0$ and $q_i^{(a,a)} = q_1^{(a,a)} = 1$ for all $i \in 2\mathbb{Z}_+ + 1$.

This proves the lemma. ∎

Lemma 2.3 shows that the diagonal elements of $P_{j,k}$ are $[\bar{k}]^{j+1}$ for $j = 1, 2$, and thus for all $j \geq 1$ since Vir_+ is generated by tD, t^2D.

LEMMA 2.4: $P_1 = P_2 = 0$ and $Q_i = 1_N$ for all $i \in 2\mathbb{Z}_+ + 1$.

Proof: For a given position (a, b) with $1 \leq a < b \leq N$, suppose inductively we have proved

$$(2.17) \qquad p_1^{(a_1,b_1)} = p_2^{(a_1,b_1)} = 0, \quad q_i^{(a_1,b_1)} = \delta_{a_1,b_1},$$

for all $i \in 2\mathbb{Z}_+ + 1$ and for $a_1 > a, b_1 \leq b$ or $a_1 \geq a, b_1 < b$. Denote now

$$p_{j,k} = p_{j,k}^{(a,a)}, \quad p_j = p_j^{(a,a)}, \quad p'_{j,k} = p_{j,k}^{(a,b)}, \quad p'_j = p_j^{(a,b)}, \quad q_i = q_i^{(a,b)},$$

for $j \in \mathbb{Z}, i \in 2\mathbb{Z}_+ + 1$, and denote

$$\widetilde{P}_{j,k} = \begin{pmatrix} p_{j,k} & p'_{j,k} \\ 0 & p_{j,k} \end{pmatrix}, \quad \widetilde{P}_j = \begin{pmatrix} 0 & p'_j \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \widetilde{Q}_i = \begin{pmatrix} 1 & q_i \\ 0 & 1 \end{pmatrix}.$$

Then these 2×2 matrices commute with each other. By assumption (2.17), we see that (2.7)–(2.9) still hold when we replace all matrices by their corresponding matrices with tilde, and we have similar formulas for $P_{j,k}, j = 3, 4, 5$ as in (2.6) and (2.14) (here now, $[\widetilde{P}_1, \widetilde{P}_2] = 0$). Since $\widetilde{Q}_i$ is invertible, from (2.8), we obtain an equation on $\widetilde{P}_{i,k}$. Using (2.5) and (2.6) in this equation, we obtain that $4[i]_3(3\widetilde{P}_1 - \widetilde{P}_2) = 0$. This shows that $\widetilde{P}_2 = 3\widetilde{P}_1$. Then (2.7) and (2.9) give

$$[i + 1]_4\widetilde{Q}_{i-2} = [i]_2(i^2 - i + 12\widetilde{P}_1 - 2)\widetilde{Q}_i,$$
$$[i + 1]_6\widetilde{Q}_{i-4} = [i]_4(i^2 - 3i + 30\widetilde{P}_1 - 4)\widetilde{Q}_i.$$

Since $\widetilde{Q}_i$ are invertible, the above gives $\widetilde{P}_1 = 0$ and so $\widetilde{P}_2 = 0$. Then the above also gives $\widetilde{Q}_i = \widetilde{Q}_1 = 1_2$ for $i \in 2\mathbb{Z}_+ + 1$. This proves that (2.17) holds for (a, b). Thus we have the lemma. ∎

Thus by Lemma 2.4 and (2.5), $P_{1,k} = [\bar{k}]^2$, $P_{2,k} = [\bar{k}]^3$, $Q_i = 1$, $i \in 2\mathbb{Z}_+ + 1$, are all scalar matrices for $k >> 0$. By shifting the grading index of V_k if necessary, we can suppose that $[\bar{k}]^2, [\bar{k}]^3 \neq 0$ and (2.4) holds for $k \geq 0$. Applying $[(\frac{d}{dt})^2, [(\frac{d}{dt})^2, t^2 \frac{d}{dt}]] = 8(\frac{d}{dt})^3$ to Y_0, we obtain that $Q_2^2 = 1$. Thus by linear algebra, Q_2 is a diagonalizable matrix. Note that

$$(2.18) \qquad \sigma = \left(\frac{d}{dt}\right)^2 \cdot \left(t^3 \frac{d}{dt}\right)|_{V_0}$$

is a linear transformation on V_0 (recall (1.2) for the product "$\cdot$"), such that $\sigma Y_0 = [\bar{2}]^3 Y_0 Q_2$. Thus by re-choosing the basis Y_0 and re-defining Y_k such that (2.4) holds for $k \geq 0$ (then this change of basis Y_k does not effect $P_{1,k}, P_{2,k}$, $Q_i, i \in 2\mathbb{Z}_+ + 1$, since they are scalar matrices), we can then suppose Q_2 is a diagonal matrix (with the diagonal elements of Q_2 being ± 1).

LEMMA 2.5: *For all $i, k \in \mathbb{Z}$ with $k, k + i \geq 0$, $P_{i,k}$ is a scalar matrix.*

Proof: Using $[tD, t^{i-1}D] = (i - 2)t^i D$ and (2.5), by induction on i, we obtain $P_{i,k} = [\bar{k}]^{i+1}$ for $i \geq -1, k \geq 0$. Thus assume that $i = -i_1 \leq -2, k + i \geq 0$. Let j be any integer such that $j > i_1$. Applying $(j + i_1)t^{j-i_1}D = [t^{-i_1}D, t^j D]$ to Y_k, we obtain

$$(j + i_1)[\bar{k}]^{j-i_1+1} = [\bar{k}]^{j+1}P_{-i_1,k+j} - [\bar{k} - i_1]^{j+1}P_{-i_1,k}.$$

By replacing k by $k + j$ and replacing j by $2j$, we obtain two other equations respectively. From these three equations, one can easily deduce that $P_{-i_1,k}$ is a scalar matrix.   ∎

Since W is generated by $\mathrm{Vir} \cup \{(\frac{d}{dt})^2\}$, by induction on j, one can prove

$$(2.19) \qquad (t^{i+j}(\frac{d}{dt})^j)Y_k = Y_{k+i}P_{i,j,k} \quad \text{for some diagonal matrices } P_{i,j,k},$$

and for all $i, j, k \in \mathbb{Z}$ with $j \geq 1, k, i + k \geq 0$.

LEMMA 2.6: *Denote by $V(a)$ the W-submodule of V generated by $y_0^{(a)}$, $a = 1, \ldots, N$. Then $V(a)$ is a module of the intermediate series such that $V' = V(1) + \cdots + V(N)$ is a direct sum of W-submodules.*

Proof: Since $U(W) = U(W_-)U(W_0 + W_+)$ and $V(a) = U(W)y_0^{(a)}$, by writing $u \in U(W)$ as a sum of $u_1 \cdots u_r w_1 \cdots w_s$ for $u_i \in W_-$, $w_i \in W_0 + W_+$, using (2.19), we obtain by induction on $r + s$ that $\dim V(a)_k = 1$ for $k \geq 0$. Since $V(a)$ is also a Vir-module, by [S2], $\dim V(a)_k = 1$ for all k with $k + \alpha \neq 0$.

Then by (2.5) and the above lemmas, one can prove that $V(a)$ is a subquotient module of A_α or B_α, i.e., $V(a)$ is a W-module of the intermediate series (also cf. [Z]).

For $a = 1, \ldots, N$, let $V'(a) = V(a) \cap \sum_{i \neq a} V(i)$. Then obviously, $V'(a)_k = \{0\}$ for $k \geq 0$. Thus we must have $V'(a) = \{0\}$. This proves the lemma. ∎

Now let $V'' = V/V'$. Then V'' is a finite dimensional trivial module. By induction on the number $N + \dim V''$, one obtains that V is decomposable if $N \geq 2$. Thus $N = 1$ and one can further deduce that V is a module of the intermediate series. This proves Theorem 1.2(i).

COROLLARY 2.7: *Suppose V is a uniformly bounded quasifinite W-module satisfying (2.2) and there exists $N \geq 1$ such that $\dim V_i = N$ for all $i \in \mathbb{Z}$ with $\alpha + i \neq 0$. Fix $i_0 \in \mathbb{Z}$ with $\alpha + i_0 \neq 0$ and fix a basis Y_{i_0} of V_{i_0}. Then there exists a basis Y_k of V_k for all $k \in \mathbb{Z}$ with $\alpha + k \neq 0$ such that $(t^j D)Y_{i_0} = (\alpha + i_0)Y_{i_0 + j}$ for all $j \in \mathbb{Z}$ with $\alpha + i_0 + j \neq 0$.* ∎

3. Quasifinite $\mathcal{W}(\Gamma, n)^{(1)}$-modules

Since Theorem 1.1(ii) is a special case of Theorem 1.2(ii), we shall prove Theorem 1.2(ii) (cf. [S4]). Thus assume that Γ is a group not isomorphic to $\mathbb{Z}$ and V is an indecomposable quasifinite weight $\mathcal{W}(\Gamma, n)^{(1)}$-module such that there exists some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$ with (cf. (2.2))

$$V_\beta = \{v \in V | D_i v = (\alpha_i + \beta_i)v, i = 1, \ldots, n\} \quad \text{for } \beta \in \Gamma.$$

As the proof in [S4], V is uniformly bounded, and there exists $N \geq 0$ such that $\dim V_\beta = N$ for all $\beta \in \Gamma$ with $\alpha + \beta \neq 0$. For convenience, we shall now denote $\bar{\mu} = \mu + \alpha$ for all $\mu \in \mathbb{F}^n$.

By [SZ2], we can suppose that all elements $\gamma(i) = (\delta_{1,i}, \ldots, \delta_{n,i})$ for $i = 1, \ldots, n$ are in Γ. We denote $\mathcal{D} = \bigoplus_{i=1}^n \mathbb{F}D_i$ and define an inner product on $\Gamma \times \mathcal{D}$ by

$$(3.1) \qquad \langle \beta, d \rangle = \sum_{i=1}^n \beta_i d_i \quad \text{for } \beta = (\beta_1, \ldots, \beta_n) \in \Gamma, \ d = \sum_{i=1}^n d_i D_i \in \mathcal{D}.$$

Then $\langle \cdot, \cdot \rangle$ is **nondegenerate** in the sense that if $\langle \beta, \mathcal{D} \rangle = 0$ for some $\beta \in \Gamma$ then $\beta = 0$ and if $\langle \Gamma, d \rangle = 0$ for some $d \in \mathcal{D}$ then $d = 0$.

By (1.2) and (3.1), we have

$$(3.2) \qquad [t^\beta d, t^\gamma d'] = t^{\beta + \gamma}(\langle \gamma, d \rangle d' - \langle \beta, d' \rangle d) \quad \text{for } \beta, \gamma \in \Gamma, \ d, d' \in \mathcal{D}.$$

Fix an element $\gamma \in \Gamma$ such that $\overline{\gamma}, \overline{\gamma} \pm \gamma(i), \overline{\gamma} \pm 2\gamma(i) \neq 0$ for $i = 1, \ldots, n$. As in (2.18),

$$\sigma_i = (t^{-2\gamma(i)} D_i (D_i - 1)) \cdot (t^{2\gamma(i)} D_i)|_{V_\gamma} \quad \text{for } i = 1, \ldots, n,$$

are diagonalizable operators (note that $(\frac{d}{dt})^2 = t^2 D(D-1)$ and $t^3 \frac{d}{dt} = t^2 D$ in (2.18)). Since $\sigma_i, i = 1, \ldots, n$, commute with each other, one can choose a basis Y_γ of V_γ such that σ_i correspond to diagonal matrices. Let $\beta \in \Gamma \backslash \{0\}$ be any element such that $\overline{\gamma} + \beta \neq 0$. We shall define a basis $Y_{\gamma+\beta}$ of $V_{\gamma+\beta}$ as follows: One can choose some $d \in \mathcal{D}$ such that $\langle \overline{\gamma}, d \rangle, \langle \beta, d \rangle, \langle \overline{\gamma} + \beta, d \rangle \neq 0$. Let $W(\beta) = \text{span}\{t^{i\beta} d^j | i \in \mathbb{Z}, j \in \mathbb{Z}_+ \backslash \{0\}\}$ be a Lie subalgebra of $\mathcal{W}(\Gamma, n)^{(1)}$, which is isomorphic to $\mathcal{W}(\mathbb{Z}, 1)^{(1)}$ by (3.2) (cf. [S4]). Denote $V(\beta) = \bigoplus_{i \in \mathbb{Z}} V_{\gamma+i\beta}$. Then $V(\beta)$ is a uniformly bounded quasifinite $W(\beta)$-module. By Corollary 2.7, $t^\beta d|_{V_\gamma} : V_\gamma \to V_{\gamma+\beta}$ is bijective. We define $Y_{\gamma+\beta} = \langle \overline{\gamma} + \beta, d \rangle^{-1} (t^\beta d) Y_\gamma$.

Now as in (2.19), one can prove by induction on $|\mu| = \mu_1 + \cdots + \mu_n$ that $(t^\beta D^\mu) Y_\eta = Y_{\eta+\beta} P_{\beta,\mu,\eta}$ for some diagonal matrices $P_{\beta,\mu,\eta}$ and for all $\beta, \eta \in \Gamma, \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_+^n \backslash \{0\}$ with $\overline{\eta}, \overline{\eta} + \beta \neq 0$. Thus as the proof of Lemma 2.6, we obtain that V must be a module of the intermediate series. This proves Theorem 1.2(ii).

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